

Nonstandard Cutoff Effects in the Nonlinear Sigma Model*

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High precision measurements of the renormalized zero-momentum 4-point coupling g_R and of the Lüscher-Weisz-Wolff running coupling $\bar{g}(L) = Lm(L)$ performed with two different lattice actions in the non-perturbative region confirm the earlier observations, that the cutoff effects look linear, in contrast to perturbative considerations. The use of different actions allows one to make a more reliable estimate on the continuum limit. The measurements were done for infinite volume correlation length up to 350.

In numerical simulations with lattice regularization the physical results are obtained after extrapolating to zero lattice spacing. Usually (and especially in 4d) the range of a values accessible in the simulations is quite restricted, therefore the extrapolation $a \rightarrow 0$ is highly non-trivial and the knowledge of the functional form of the artifacts is essential. As shown by Symanzik [1], in bosonic theories in every order of perturbation theory (PT) the leading lattice artifacts decrease like $O(a^2)$ (apart from $\log a$ factors). It is generally assumed that this behaviour holds beyond PT, and the extrapolations in numerical simulations are done according to this assumption.

In [2] the renormalized zero-momentum 4-point coupling was measured in the $O(3)$ non-linear sigma model to high precision. Unexpectedly, the cutoff effects were nearly linear in a and were described well by a form $c_0 + c_1a + c_2a \log a$, in contrast to the standard belief. (Note that already in [3] there was some numerical evidence that a linear Ansatz describes the data better, but the errors were significantly larger.)

Here we extend the results of [2] to another quantity, the Lüscher-Weisz-Wolff running coupling $\bar{g}(L)$ [4]. We also measure these quantities using an alternative (ad hoc) lattice action containing a diagonal interaction term. For this action the coefficient of the nearest neighbour (on-

axis) term is β_1 while the coefficient of the diagonal term is β_2 . They were chosen to be equal, $\beta_1 = \beta_2 = \beta/3$. (In this notation for the standard action $\beta_1 = \beta$, $\beta_2 = 0$.)

The simulations were done with both actions at the same values of the correlation length and lattice size (i.e. same a/L and $\xi(L)$). The assumption of universality implies that both actions lead to the same continuum limit for physical quantities. This strongly restricts the fits to the data.

The quantities measured are:

1) The renormalized 4-point coupling at finite volume

$$g_R(z) = \left(\frac{L}{\xi_2(L)} \right) \left(\frac{5}{3} - \frac{\langle (\mathbf{M}^2)^2 \rangle}{\langle \mathbf{M}^2 \rangle^2} \right), \quad (1)$$

defined on an $L \times L$ periodic lattice. Here $\mathbf{M} = \sum_x \mathbf{S}$ is the total magnetization while $\xi_2(L)$ is the second-moment correlation length (cf. e.g. [3]). The physical size, $z = L/\xi_2(L)$ is kept fixed.

2) The running of the LWW coupling [4]. The coupling $\bar{g}(L) = m(L)L$ is defined on an $L \times \infty$ strip, where $m(L)$ is the inverse correlation length in the zero-momentum spin-spin correlation function. For a given number L/a of lattice sites in the spatial direction one tunes the value of β to get a fixed value $\bar{g}(L) = u_0$. In the next step one measures $\bar{g}(2L)$ at the same value of β , giving $\bar{g}(2L) = \Sigma(2, u_0, a/L)$ [4]. The physical (regularization independent) “step scaling function” is given by the

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L/a	(st)	(mod)
10	3.0105(1)	2.9237(1)
28	3.0765(2)	3.0469(1)
56	3.0979(2)	3.0844(1)
112	3.1095(2)	3.1032(1)
224	3.1145(3)	3.1119(1)
448	3.1175(6)	3.1159(2)

Table 1

$g_R(z_0)$ for the two actions at $z_0 = 2.32$.

L/a	(st)	(mod)
5	1.29379(8)	1.33591(12)
10	1.27994(9)	1.29706(14)
12	1.27668(9)	1.29060(13)
16	1.27228(12)	1.28242(21)
24	1.26817(9)	1.27424(10)
32	1.26591(9)	1.27014(15)
64	1.26306(16)	1.26456(20)

Table 2

$\Sigma(2, u_0, a/L)$ at $u_0 = 1.0595$.

limit $\lim_{a \rightarrow 0} \Sigma(2, u_0, a/L) = \sigma(2, u_0)$. Due to an improved estimator by Hasenbusch [5] (suited for the strip geometry) we could significantly improve the error.

The values of $g_R(z_0)$ at $z_0 = 2.32$ for different lattices sizes L/a and the two actions are given in Table 1. These data are shown in Figure 1. Table 2 and Figure 2 refer to the LWW step scaling function $\Sigma(2, u_0, a/L)$ at $u_0 = 1.0595$.

The data with $L/a > 10$ were fitted simultaneously for the two actions by several forms:

$$\begin{aligned}
\text{A: } & c_0 + c_1 \frac{a}{L} + c_2 \frac{a}{L} \log \frac{L}{a}, \\
\text{B: } & c_0 + c_1 \left(\frac{a}{L}\right)^2 + c_2 \left(\frac{a}{L}\right)^2 \log \frac{L}{a}, \\
\text{C: } & c_0 + c_1 \left(\frac{a}{L}\right)^{c_2}.
\end{aligned} \quad (2)$$

The coefficients c_0, c_1, c_2 correspond to the standard action while those for the modified action we denote by c'_0, c'_1, c'_2 . Universality implies $c'_0 = c_0$, and for the case C we also took the same power, $c'_2 = c_2$. Table 3 gives the coefficients and

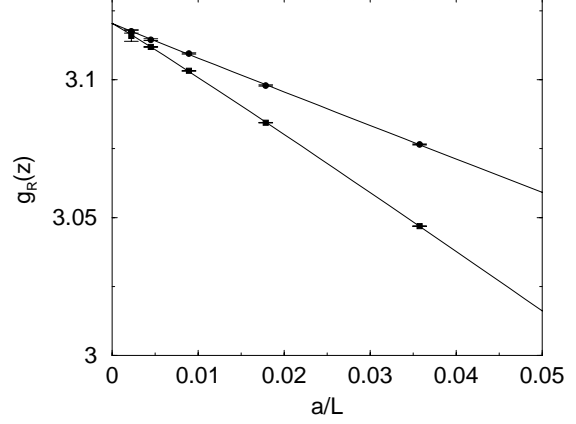


Figure 1. $g_R(z)$ at $z = 2.32$. The data for the larger artifacts correspond to the modified action. The fit shown has a and $a \log a$ terms.

the values of χ^2/dof for the fits. (The pure a^2 fits have $\chi^2/\text{dof} = 11800/7$ and $1270/7$ hence are not listed.)

Interestingly, one does not necessarily need to assume a specific analytic form of the cutoff dependence. Since we performed the measurements with the two actions at the same values of a/L and $\xi(L)$ it is enough to assume that the form of the leading cut-off effect is the same, only the prefactors are different. A quantity q measured with the two actions is given then by

$$\begin{aligned}
q(a) &= q_0 + f(a) + \dots \\
q'(a) &= q_0 + cf(a) + \dots
\end{aligned} \quad (3)$$

where $f(a)$ is an unspecified function. We assume that at the values of a considered the non-leading artifacts are negligible. Measuring the quantity q at different values a_1, a_2, \dots, a_N one has $2N$ data $q(a_i), q'(a_i)$ for the two actions and only $N+2$ unknown variables in the fit, q_0, c and $f(a_i)$. Both $g_R(z_0)$ and $\sigma(2, u_0)$ are described well by the assumptions made in (3). In Table 4 we give the continuum value, c and χ^2/dof for the two quantities measured.²

For small $\bar{g}(L)$ the cutoff effects can be studied in PT. The sign of artifacts at $\bar{g}(L) = O(1)$

²Obviously, by taking three different actions one can resolve the next-to-leading cutoff term as well.

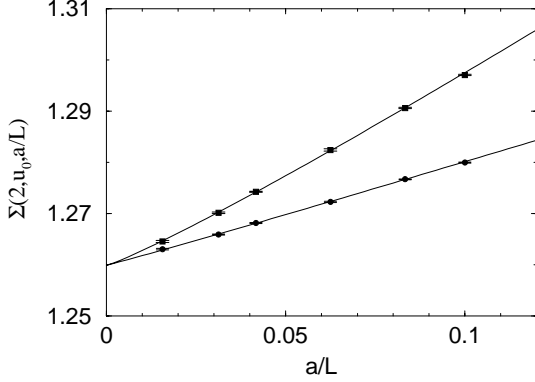


Figure 2. The step scaling function $\Sigma(2, u_0, a/L)$ at $u_0 = 1.0595$. The data for the larger artifacts correspond to the modified action. The fit contains a and $a \log a$ terms.

(like in Fig. 2) is opposite to that obtained in PT. This has been noticed already in [4], where it has also been pointed out that the large N limit shows the same effect. Caracciolo et al. [6] studied the large N integrals analytically and found $(a/L)^2 (\log L/a)^{-q}$ artifacts with $q = -1, 0, 1, 2, \dots$ in the step scaling function. They pointed out that the negative powers of $\log L/a$ are not reproduced by PT.

Based on our numerical study in $O(3)$ it can not be ruled out that for very small a the perturbative prediction (a^2 times powers of $\log a$) takes over. In this case an explanation would be needed why this happens at $\xi > 350$ only.

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fit	c_0	c_1 c'_1	c_2 c'_2	χ^2/D
A	3.12121(7)	-1.266(6) -2.075(4)	—	86/7
A	3.1204(2)	-1.17(6) -2.32(4)	-0.02(2) 0.08(2)	18/5
B	3.11568(9)	87(3) 165(2)	-35(1) -65.8(6)	180/5
C	3.1201(2)	-1.42(3) -2.36(5)	1.043(7) = c_2	44/6
A	1.2591(1)	0.213(2) 0.372(2)	—	74/7
A	1.2599(4)	0.22(2) 0.46(2)	-0.007(8) -0.038(8)	5.2/5
B	1.2619(2)	-2.9(3) -6.5(3)	2.0(1) 4.3(1)	11/5
C	1.2607(2)	0.28(1) 0.51(2)	1.14(2) = c_2	18/6

Table 3

Fits of the form (2) for $g_R(z_0)$ and $\sigma(2, u_0)$.

	q_0	c	χ^2/D
$g_R(z_0)$	3.1181(5)	1.71(1)	6.7/3
$\sigma(2, u_0)$	1.2614(3)	1.91(2)	1.0/3

Table 4

Results of the form-independent fits.

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